An estimation-free, robust CVaR portfolio allocation model

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Abstract

We propose a novel optimization model to obtain robust solutions for portfolio allocation problems. Unlike related models in the literature, no historical data or statistical estimation techniques are used to compute the parameters of the model. Instead, the parameters are directly obtained from current prices of options on the assets being considered. Furthermore, the model only requires the solution of a linear program. To find a robust portfolio, we minimize the portfolio's worst-case Conditional Value-at-Risk over all assets return distributions that replicate the current option prices. The model addresses the main practical limitations associated with classical portfolio allocation techniques; namely, the high sensitivity to model parameters, and the difficulty to obtain accurate parameters' estimates. These characteristics, together with its linear programming formulation, and the use of a coherent downside measure of risk, should be appealing to practitioners. We provide numerical experiments to illustrate the characteristics of the model.

1 Introduction

The problem of portfolio allocation; that is, deciding the optimal (best) way to invest available funds into different assets, is one of the central problems in today's business world. Starting with the seminal work of Markowitz [20], a number of *classical* optimization-based models have been proposed and investigated to solve the portfolio allocation problem (see, e.g., [12] for a recent review of such models). In the Markowitz mean-variance model [20], the optimal portfolio is found by minimizing the variance of the portfolio returns over all the portfolios with a given expected return. Although the development of these models have produced great theoretical impact, their practical relevance has been limited, especially for those which are based on Markowitz mean-variance model. The main reason behind this well-documented fact (see, e.g., [16, 21] and the references therein) is the lack of *robustness* of these classical portfolio allocation models; that is, optimal portfolios from these models are very sensitive to changes in the model's parameters (see, e.g., [5, 9, 10, 13]). Moreover, these parameters are well-known to be difficult to estimate accurately (see, e.g., [10, 13, 21]). This combination of sensitivity and estimation error results in optimal portfolios that are very unreliable. In particular, mean-variance based portfolios are considered to be (quoting Michaud [21]) "[...]*error maximized*, and have little, if any, reliable investment value." Typically, the parameters include the mean and variance-covariance of the assets' returns, for *static* portfolio models; and the parameters driving the underlying processes of the riskfree interest rate, and the asset returns, for *dynamic* portfolio models (cf. [12]). Their estimation is done by using historical data, and a variety of statistical techniques (cf. [12, 21]).

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The development of models to obtain robust solutions to portfolio allocation problems has been prompted by these practical limitations in classical models, and advances in the areas of robust optimization (see, e.g., [3]), and robust statistics (see, e.g., [18]). For example, consider the work of Ben-Tal, Margalit, and Nemirovski [2]; Black and Litterman [6]; Ceria and Stubbs [9]; Cavadini, Sbuelz, and Trojani [7]; DeMiguel and Nogales [13]; El Ghaoui, Oks, and Oustry [14]; Erdogan, Goldfarb, and Iyengar [15]; Goldbarb and Iyengar [16]; Lu [19]; Michaud [21] (and the references therein); Perret–Gentil and Victoria–Feser [23]; Pinar and Tütüncü [24]; Tütüncü and Koenig [27]; and Zhu and Fukushima [29]; to name just a few.

We propose a novel optimization model to obtain robust solutions for portfolio allocation problems. Unlike related models in the literature, no historical data or statistical estimation techniques are used to compute the parameters of the model. Instead, the parameters of the model are directly obtained from current prices of European forward and call options on the assets being considered. Notice that current option prices are the result of both historical market trend estimations, and special current market events; which are typically not captured by historical estimations. These prices also reflect a consensus of investor's views rather than a single investor's view of the market.

The proposed portfolio allocation model only requires the solution of a linear program. This contrasts with other related models; which require the use of more involved optimization techniques. For example, the models in [14, 15, 16, 19, 24] use advanced optimization techniques; namely semidefinite, and second-order cone programming, for which solvers are not nearly as available and developed as linear programming solvers.

To obtain an optimal robust portfolio, we minimize the portfolio's worst-case Conditional Value-at-Risk (cf. [25]) over all assets return distributions that replicate the current option prices (see Section 2). Thanks to its properties (see, e.g., [26]), Conditional Value-at-Risk (CVaR) has become one of the most popular risk measures among researchers, and keeps gaining popularity among practitioners.

Our approach is related to the recent work of El Ghaoui et al. [14]; Natarajan, Pachamanova, and Sim [22]; and Zhu and Fukushima [29]. To obtain robust portfolios, these authors also consider a worst-case CVaR (or worst-case Value-at-Risk in [14]) in the case where only partial information on the assets return distribution is given. The main difference of our approach is the partial information considered here: We use information about option prices on the assets instead of *moments* (means, and variance-covariances). This particular choice of partial information is the key behind the characteristics of our model.

As it will be illustrated by numerical experiments (see Section 3), our model addresses the main practical limitations associated with classical portfolio allocation techniques; namely, the high sensitivity to model parameters, and the difficulty to obtain accurate parameters' estimates. These characteristics, together with its linear programming formulation, and the use of CVaR; a coherent downside measure of risk (see, e.g., [26]), should be appealing to practitioners. This should be especially true for *passive* and/or *risk-averse* investors (i.e., who want to avoid constant rebalancing, and/or are typically interested in avoiding high losses), or investors without full access to historical data, and advanced statistical estimation or optimization techniques. This type of model should also be appealing to practitioners in Actuarial Science, who face portfolio allocation problems over assets for which very sparse historical data is available (e.g., sometimes only quarterly data is available).

It is worth mentioning that in the context of portfolio allocation, robustness is also used to describe the out-of sample (i.e., "future") performance of the optimal portfolios (cf. [12]). A portfolio is considered to be robust, if instead of giving the best performance (return) in certain out-of sample scenarios of the market, it gives a good or fair performance under as many out-of sample scenarios as possible. This can be measured using the out-of-sample Sharpe ratio of the portfolio returns (see, e.g. [12]). The higher the Sharpe ratio of a portfolio, the more robust it is considered. Also, robustness is measured by the amount of rebalancing that is necessary to maintain the portfolio strategy over time. The lower the amount of rebalancing, the more robust the strategy is (see, e.g. [12]). Although the numerical experiments presented here focus on empirically showing robustness of our model in terms of its sensitivity to input parameters, the worst-case approach adopted in the model aims also at obtaining portfolios with robust out-of-sample performance. However, obtaining conclusive empirical evidence of the out-of-sample performance of a portfolio allocation model requires comprehensive experiments (see, e.g., [12]) that are outside the scope of this article, and will be the subject for future work.

The article is organized as follows. In Section 2, we present the proposed robust portfolio allocation model, and show that it can be formulated as a linear program. In Section 3, we illustrate the properties of our model by presenting some numerical results. Specifically, we use information about index options to address the problem of strategically allocating funds among different asset classes. In Section 4, we consider some extensions to the model, and present how these extensions can be used to rebalance a portfolio composed by the Dow Jones 30 Index assets.

2 The new portfolio allocation model

Consider n risky assets that can be chosen by an investor in the financial market. Let $r = (r_1, \ldots, r_n)^{\mathrm{T}} \in \mathrm{IR}^n$ denote the uncertain returns of the n risky assets from the current time $t = 0$ to a fixed future time $t = T$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathrm{IR}^n$ denote the percentage of the available funds to be allocated in each of the n risky assets. A (static) portfolio allocation model aims at finding the optimal (best) portfolio x to be constructed at $t = 0$, in order to maximize the portfolio's future return $r^{\mathrm{T}}x$ from $t = 0$ to $t = T$.

2.1 Nominal portfolio allocation model

We begin by introducing the measure of risk that will be the basis of the proposed portfolio allocation model; namely, Conditional Value-at-Risk (CVaR); which measures the shortfall of the portfolio returns distribution (cf. [25]). Specifically, following [25], we define the β-level CVaR as

$$\mathrm{CVaR}_\beta(f(r,x)) := \mathbb{E}(f(r,x) | f(r,x) \geq \alpha^*(\beta)),$$

where $f(r,x)$ is the portfolio loss function, $\beta \in (0,1)$, and α^* is the portfolio's β-level Value-at-Risk (cf. [25]) defined by

$$\alpha^*(\beta) := \min\{\alpha \in \mathrm{IR} : \mathbb{P}(f(r,x) \geq \alpha) \geq \beta\}.$$

Here, $\mathbb{E}(\cdot)$ and $\mathbb{P}(\cdot)$ denote respectively the expectation and probability over the assets return distribution. Thus, the portfolio CVaR is the expected value of the β-quantile of the portfolio loss distribution.

Typically (see [25]), the portfolio loss function is defined as the negative of the portfolio return (cf. [25]); that is,

$$f(r,x) = -r^{\mathrm{T}}x,$$

and $0.90 \leq \beta \leq 0.99$.

CVaR has become one of the most popular risk measures among researchers, and keeps gaining popularity among practitioners. This is due to both its theoretical and practical properties. For example, in contrast with the variance of the portfolio returns (used in Markowitz's mean-variance model (cf. [20]) as the measure of risk), the CVaR is a coherent downside measure of risk (cf. [1, 26]).

Now we introduce the following *nominal* portfolio allocation model:

$$\begin{array}{ll} \min\limits_{x \in \mathcal{X} \subseteq \mathrm{IR}^n_+} & \mathrm{CVaR}_\beta(-r^{\mathrm{T}}x) \\ \text{s.t.} & e^{\mathrm{T}}x = 1, \end{array} \tag{1}$$

where e is the vector of all-ones, and $\mathcal{X} \subseteq \mathrm{IR}^n_+$ is a set defined by linear constraints; which are typically used to enforce certain diversification constraints on the portfolio x. For a given $\beta \in (0,1)$, the model (1) looks for the portfolio that minimizes the β-level CVaR of the portfolio losses $(-r^{\mathrm{T}}x)$, over all admissible portfolios $x \in \mathcal{X}$. As discussed before, β defines the quantile of the portfolio loss distribution. For the moment, we will concentrate on portfolios with no short positions (i.e., $x \in \mathrm{IR}^n_+$). In Section 4.1, we will extend the results to portfolios in which short positions are allowed.

Notice that in the nominal portfolio allocation model (1), there is no minimum requirement on the expected portfolio return; a constraint that is present in most portfolio allocation models. In Section 4.3, we will show how this constraint can be added to our proposed portfolio allocation model, while maintaining all the properties of the model. The reasons behind leaving the minimum expected return constraint out for now, are the recent results of DeMiguel and Nogales [13]; which show that leaving out this constraint typically leads to portfolios that are more robust (see, [13] for a detailed discussion).

2.2 Robust portfolio allocation model

In order to introduce the proposed robust version of (1), we first need to express the asset returns in terms of the asset prices. Let $S_{t,i}$, $i = 1, \dots, n$, denote the price of the i-th risky asset at time $t \geq 0$. Furthermore, let $S_0 = (S_{0,1}, \dots, S_{0,n})^{\mathrm{T}} \in \mathbb{R}^n_{++}$ denote the known current ($t = 0$) prices of the n risky assets, and $S_T = (S_{T,1}, \dots, S_{T,n})^{\mathrm{T}} \in \mathbb{R}^n_{+}$ denote the uncertain prices of the n risky assets at the future time $t = T$. Notice that

$$r_i = \frac{S_{T,i} - S_{0,i}}{S_{0,i}} = \frac{S_{T,i}}{S_{0,i}} - 1, \tag{2}$$

for $i = 1, \dots, n$. Therefore,

$$r^{\mathrm{T}} x = \sum_{j=1}^{n} \left(\frac{S_{T,i}}{S_{0,i}} \right) x_i - 1 := \hat{s}^{\mathrm{T}} x - 1,$$

where we have used the fact that $e^{\mathrm{T}} x = 1$, and defined

$$\hat{s} := \left(\frac{S_{T,1}}{S_{0,1}}, \dots, \frac{S_{T,n}}{S_{0,n}} \right)^{\mathrm{T}}. \tag{3}$$

With this, we can now restate the nominal portfolio model (1) as:

$$\begin{array}{ll} \min\limits_{x \in \mathcal{X} \subseteq \mathbb{R}^n_+} & \mathrm{CVaR}_\beta(1 - \hat{s}^{\mathrm{T}} x) \\ \text{s.t.} & e^{\mathrm{T}} x = 1, \end{array} \tag{4}$$

where the expectations and probabilities are now taken over the assets price distribution at *maturity* ($t = T$), instead of the assets return distribution, based on the relationship (2). To construct a robust version of (4), we minimize the worst-case portfolio CVaR over all assets price distributions at maturity that *replicate* current prices of European forward and call options on the assets. Specifically, we introduce the following robust formulation of (4):

$$\begin{array}{lll} \min\limits_{x \in \mathcal{X} \subseteq \mathbb{R}^n_+} & \sup\limits_{\pi(S_T) \in \mathcal{P}} & \mathrm{CVaR}_{\beta, \pi(S_T)}(1 - \hat{s}^{\mathrm{T}} x) \\ \text{s.t.} & & e^{\mathrm{T}} x = 1, \end{array} \tag{5}$$

where $\pi(S_T)$ represents the assets price distributions at maturity. Given $p^0 \in \mathbb{R}^n$, and $K^j, p^j \in \mathbb{R}^n$, $j = 1, \dots, m$, the *uncertainty* set $\mathcal{P}$ is defined by

$$\mathcal{P} = \left\{ \begin{array}{ll} \pi(S_T): & \mathbb{E}_{\pi(S_T)}(1) = 1 \\ & \mathbb{E}_{\pi(S_T)}(S_T) = p^0 \\ & \mathbb{E}_{\pi(S_T)}((S_T - K^j)) = p^j, j = 1, \dots, m \\ & \pi(S_T) \text{ is a distribution in } \mathbb{R}^n_+ \end{array} \right\}. \tag{6}$$

The definition of the uncertainty set (6) is motivated by the *arbitrage bounds* literature (see, e.g., [11, 17, 28] and the references therein). Specifically, in (6), p_i^0 is the given price of a European forward option on asset i maturing at $t = T$, for $i = 1, \dots, n$; and p_i^j is the given price of a European call option on asset i with strike K_i^j maturing at $t = T$, for $j = 1, \dots, m$, $i = 1, \dots, n$. The assumption of having information about the same number m of call options per asset is done only to simplify the exposition.

Throughout the article, we will assume that the given prices are *arbitrage-free* (cf. [4]); that is, they satisfy the following convexity condition:

$$0 \leq \frac{p_i^{m-1} - p_i^m}{K_i^m - K_i^{m-1}} \leq \frac{p_i^{m-2} - p_i^{m-1}}{K_i^{m-1} - K_i^{m-2}} \leq \dots \leq \frac{p_i^1 - p_i^2}{K_i^2 - K_i^1} \leq \frac{p_i^0 - p_i^1}{K_i^1} \leq 1, \tag{7}$$

for $i = 1, \dots, n$.

Notice that in (5) we have made explicit the dependence of the portfolio CVaR on the assets price distribution (at maturity).We can rewrite (5) as

$$\begin{array}{ll} \min\limits_{x \in \mathcal{X} \subseteq \mathbb{R}^n_+} & \mathrm{WCVaR}_\beta(1 - \hat{s}^{\mathrm{T}} x) \\ \text{s.t.} & e^{\mathrm{T}} x = 1, \end{array} \tag{8}$$

where similar to [22, 29], we define the β-level *worst-case* CVaR (WCVaR) as:

$$\mathrm{WCVaR}_\beta(1-\hat{s}^{\mathrm{T}}x) := \sup_{\pi(S_T)\in\mathcal{P}} \mathrm{CVaR}_{\beta,\pi(S_T)}(1-\hat{s}^{\mathrm{T}}x). \tag{9}$$

The approach used to obtain the robust portfolio model (8) is related to other robust portfolio allocation models in the literature; such as [14, 22, 29]. The main difference is the definition of the uncertainty set in terms of option prices. As will be shown in the next section, this definition is the key behind the characteristics of the model.

2.3 LP formulation of the model

To obtain a linear programming (LP) formulation of (8), the key is to show that (9) admits a LP formulation. In their seminal work, Rockafellar and Uryasev [25] show that

$$\mathrm{CVaR}_\beta(-r^{\mathrm{T}}x) = \min_{\alpha\in\mathbb{R}} F_\beta(-r^{\mathrm{T}}x,\alpha), \tag{10}$$

with

$$F_\beta(-r^{\mathrm{T}}x,\alpha) := \alpha + (1-\beta)^{-1}\mathbb{E}((-r^{\mathrm{T}}x-\alpha)^+), \tag{11}$$

where $(a)^+ := \max\{a,0\}$; and as before, the expectation is taken over the assets return distribution. Rockafellar and Uryasev [25] use this formulation of CVaR to show that: if the assets return distribution is estimated by an atomic distribution (typically given by historical data), then the portfolio CVaR can be efficiently minimized using linear programming.

Using (2, 3), we can express (11) in terms of the assets price distribution at maturity:

$$F_\beta(-r^{\mathrm{T}}x,\alpha) = F_{\beta,\pi(S_T)}(1-\hat{s}^{\mathrm{T}}x,\alpha) = \alpha + (1-\beta)^{-1}\mathbb{E}_{\pi(S_T)}((1-\hat{s}^{\mathrm{T}}x-\alpha)^+), \tag{12}$$

where again, we made explicit the dependence on the assets price distribution. Thus, putting together (9, 10, 12) we have that

$$\mathrm{WCVaR}_\beta(1-\hat{s}^{\mathrm{T}}x) = \sup_{\pi(S_T)\in\mathcal{P}} \min_{\alpha\in\mathbb{R}} F_{\beta,\pi(S_T)}(1-\hat{s}^{\mathrm{T}}x,\alpha).$$

As the following proposition states, under mild conditions, the order of the sup and min above can be exchanged, thanks to the finiteness of the supremum of $F_{\beta,\pi(S_T)}$ over all $\pi(S_T)\in\mathcal{P}$, and the convexity of $F_{\beta,\pi(S_T)}$ on α (see, e.g., [8, 25]).

Proposition 1 *Let $\mathcal{P}$ be given by (6). If $\mathcal{P}\neq\emptyset$, then*

$$\mathrm{WCVaR}_\beta(1-\hat{s}^{\mathrm{T}}x) = \min_{\alpha\in\mathbb{R}} \sup_{\pi(S_T)\in\mathcal{P}} F_{\beta,\pi(S_T)}(1-\hat{s}^{\mathrm{T}}x,\alpha).$$

From Proposition 1, and using (12), it follows that

$$\mathrm{WCVaR}_\beta(1-\hat{s}^{\mathrm{T}}x) = \min_{\alpha\in\mathbb{R}} \quad \alpha + (1-\beta)^{-1} \sup_{\pi(S_T)\in\mathcal{P}} \mathbb{E}_{\pi(S_T)}(((1-\alpha)-\hat{s}^{\mathrm{T}}x)^+).$$

Notice that for any $a,b\in\mathbb{R}$, $(a-b)^+ = (b-a)^+ + (a-b)$, and for any $\pi(S_T)\in\mathcal{P}$, $\mathbb{E}_{\pi(S_T)}(S_{T,i}) = p_i^0$, $i=1,\dots,n$. Therefore,

$$\mathbb{E}_{\pi(S_T)}((1-\alpha)-\hat{s}^{\mathrm{T}}x) = (1-\alpha) - \hat{p}^{\mathrm{T}}x,$$

where we have defined

$$\hat{p} := \left(\frac{p_1^0}{S_{0,1}},\dots,\frac{p_n^0}{S_{0,n}}\right)^{\mathrm{T}}. \tag{13}$$

Thus we have that

$$\mathrm{WCVaR}_\beta(1-\hat{s}^{\mathrm{T}}x) = \min_{\alpha\in\mathbb{R}} \quad \alpha + (1-\beta)^{-1}\left\{(1-\alpha) - \hat{p}^{\mathrm{T}}x + \sup_{\pi(S_T)\in\mathcal{P}} \mathbb{E}_{\pi(S_T)}((\hat{s}^{\mathrm{T}}x-(1-\alpha))^+)\right\}. \tag{14}$$

The rightmost term in (14) is equivalent to an *arbitrage bound* on a European *basket* option (cf. [11, 17, 28]). In particular, as the next theorem states, from [28, Theorem 8, and Remark 9] it follows that this arbitrage bound can be solved by a simple linear program.

Theorem 1 ([28]) *Let $\mathcal{P}$ be given by* (6)*, and $x \in \mathbb{R}^n_+$. If the arbitrage-free condition (7) holds, then* $\sup_{\pi(S_T)\in\mathcal{P}} \mathbb{E}_{\pi(S_T)}((\hat{s}^{\mathrm{T}}x - (1-\alpha))^+)$ *is equal to*

$$
\begin{array}{ll}
\min & y \\[1ex]
\text{s.t.} & y \geq x^{\mathrm{T}}\nu(\tau_{ij}) - \tau_{ij}(1-\alpha), \qquad i = 1,\dots,n, \quad j = 1,\dots,m \\
& y \geq x^{\mathrm{T}}\nu(0) \\
& y \geq x^{\mathrm{T}}\nu(1) - (1-\alpha) \\
& y \in \mathbb{R}
\end{array}
$$

where

$$
\nu(\tau)_i = \left(\frac{1}{S_{0,i}}\right) \cdot \min_{j=0,\dots,m} \left\{ p_i^j + \tau K_i^j \right\}, \tag{15}
$$

for $i = 1,\dots,n$; and

$$
\tau_{ij} = \frac{p_i^{j-1} - p_i^j}{K_i^j - K_i^{j-1}}, \tag{16}
$$

for $i = 1,\dots,n$, $j = 1,\dots,m$.

Proof. Follows from [28, Theorem 8, and Remark 9]. □

Using Theorem 1, and putting together (8) and (14), we obtain a LP formulation of the robust portfolio allocation model (5). We state this LP formulation in the following proposition.

Proposition 2 *Let $\mathcal{P}$ be given by* (6)*, and $\mathcal{X} \subseteq \mathbb{R}^n_+$ be a set defined by linear inequalities. If the arbitrage-free condition (7) holds, then solving the robust portfolio optimization model*

$$
\begin{array}{lll}
\displaystyle\min_{x\in\mathcal{X}\subseteq\mathbb{R}^n_+} & \displaystyle\sup_{\pi(S_T)\in\mathcal{P}} & \mathrm{CVaR}_{\beta,\pi(S_T)}(1 - \hat{s}^{\mathrm{T}}x) \\
\text{s.t.} & & e^{\mathrm{T}}x = 1,
\end{array}
$$

is equivalent to solving the following linear program

$$
\begin{array}{ll}
\min & y - \hat{p}^{\mathrm{T}}x - \beta\alpha \\[1ex]
\text{s.t.} & y \geq x^{\mathrm{T}}\nu(\tau_{ij}) - \tau_{ij}(1-\alpha), \qquad i = 1,\dots,n, \quad j = 1,\dots,m \\
& y \geq x^{\mathrm{T}}\nu(0) \\
& y \geq x^{\mathrm{T}}\nu(1) - (1-\alpha) \\
& e^{\mathrm{T}}x = 1 \\[1ex]
& x \in \mathcal{X} \subseteq \mathbb{R}^n_+, y \in \mathbb{R}, \alpha \in \mathbb{R}
\end{array} \tag{17}
$$

where $\nu(\tau)_i$ is given by (15)*, for $i = 1,\dots,n$; and τ_{ij} is given by* (16)*, for $i = 1,\dots,n$, $j = 1,\dots,m$.*

Proof. Follows from Theorem 1, putting together (8) and (14), and disregarding some constants. □

Some of the properties of our robust portfolio allocation model are evident from Proposition 2. Specifically, the model can be solved using linear programming; an optimization technique for which commercial software is readily available. Moreover, the high development of commercial linear programming solvers, means that any practical portfolio allocation problem (say from tens to thousands of assets) can be quickly (in a few seconds or minutes) solved on any normal computer. This contrasts with other related models; which require the use of more involved optimization techniques. For example, the models in [14, 15, 16, 19, 24] use advanced optimization techniques such as semidefinite and second-order cone programming, for which solvers are not nearly as available and developed as linear programming solvers. In particular, semidefinite programming solvers are known to be prone to numerical stability problems. Also, notice that no statistical estimation technique, or gathering of historical data is used to build the robust portfolio allocation model. Instead, the parameters of the model (other than β) are obtained from current market data about the asset prices, and option prices on those assets. Notice that current option prices are the result of both historical market trend estimations, and special current market events; which are typically not captured by historical estimations. These prices also reflect a consensus of investor's views rather than a single investor's view of the market.

This portfolio allocation model should be especially appealing for *passive* and/or *risk-averse* investors (i.e., who want to avoid constant rebalancing, and/or are typically interested in avoiding high losses), or investors without full access to historical data, and advanced statistical estimation or optimization techniques. This type of model should also be appealing to practitioners in Actuarial Science, who face portfolio allocation problems over assets for which very sparse historical data is available (e.g., sometimes only quarterly data is available).

In the following sections, we will illustrate the above mentioned properties, and give evidence of the robustness of the model by performing some numerical experiments. Also, we will discuss extensions of the model; such as considering short (instead of only long) positions (see Section 4.1); and the possible use of American (instead of European) option prices (see Section 4.2).

3 Numerical experiments

In this section we present a simple, yet relevant numerical experiment to illustrate the main characteristics of the Robust CVaR portfolio allocation model introduced in Section 2:

- the model does not require gathering of historical data or the use of statistical estimation techniques
- the model is robust in terms of the sensitivity to the model's parameters
- the model can be solved with a linear programming solver.

For this purpose, consider the decision of strategically dividing capital among the following main asset classes: very large-cap companies, large-cap companies, mid-cap companies, small-cap companies, and bonds. To apply the Robust CVaR portfolio allocation model of Section 2 to this problem, in Table 1, we first select an Index tracking the performance of each of these assets classes.

Table 1: Tracking indices selected for each of the asset classes.

Asset Class	Index Name	Index Ticker
Very large-cap	S&P 100 Index	OEX
Large-cap	S&P 500 Index	SPX
Mid-cap	S&P Midcap 400 Index	MID
Small-cap	Russell 2000 Index	RUT
Bonds	CBOE Treasury Yield Option	TYX

Next, we collect "today's" quotes on European forward and call option prices on the indices in Table 1 maturing on a same "future" date, until which the portfolio allocation is planned to be held. We also collect information about the index prices today. For the numerical experiments presented here, we assume that the allocation of funds is being made on December 1st, 2004, and the maturity date of the options is selected to be March 19th, 2005. The corresponding option prices and index prices data is presented in Table 2 (source: Bloomberg, and WRDS).

From Table 2 we obtain the values of S_0, p^j, and K^j required in the formulation of the model (17). Given a benchmark portfolio $\bar{x} \in \mathbb{R}^n_+$, and a maximum allowable deviation from the benchmark $\delta \in \mathbb{R}_+$, for the set $\mathcal{X}$ in (17), we consider:

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n_+ : (1-\delta)\bar{x} \leq x \leq (1+\delta)\bar{x} \right\}. \tag{18}$$

That is, as it is commonly done in practical portfolio asset allocation, we consider rebalancing a benchmark portfolio, with limits on the maximum possible deviation from the benchmark. Here, we take as

Table 2: European option prices from December 1st, 2004 on contracts expiring on March 19th, 2005. The table gives forward prices and mid-market prices on European call options on December 1st, 2004 for the indices in Table 1. For every index, the first row corresponds to the different strike prices, and the second row corresponds to the option prices. The entry 0 for each index gives the forward option price. The price below each index corresponds to the index price (value) on December 1st, 2004.

Index Ticker	Option prices and strikes									
OEX	0	555	560	565	570	575	580	590	595	600
566.21	483.25	17.05	13.65	10.60	8.15	5.80	4.25	2.20	1.55	1.05
SPX	0	700	1175	1180	1200	1225	1235	1250	1275	1300
1191.37	1177.78	489.50	30.40	27.20	16.30	7.80	5.50	3.25	1.20	0.50
MID	0	630	640							
645.68	544.43	23.10	16.55							
RUT	0	650	660	670	680					
643.68	465.77	12.40	8.45	5.45	3.30					
TYX	0	45	53							
50.27	43.95	5.70	0.45							

a benchmark portfolio $\bar{x}$, the myopic "$1/n$" portfolio; that is, allocating the same capital in each asset class. Also, we set $\delta = 0.75$. The reason behind this choice (besides its obvious simplicity) are the recent results of DeMiguel, Garlappi, and Uppal [12]; which surprisingly, but convincingly show that this myopic strategy is more *robust* (in terms of its out-of-sample performance (cf. [12])), than many classical portfolio allocation models. Finally, we choose β (the quantile level of the CVaR risk measure) to be 0.95.

With all the parameters defined we now solve the LP formulation of the model (17) using `MATLAB`'s linear programming solver `LINPROG`. The resulting optimal portfolio is presented in Table 3.

Table 3: Rebalancing a benchmark "$1/n$" portfolio using the Robust CVaR portfolio allocation model (17) with $\beta = 0.95$, $\mathcal{X}$ as in (18), and maximum rebalancing of 15% ($\delta = 0.75$, $\bar{x} = (1/5)e$ in (18)). The last column indicates if the rebalancing has reached the maximum possible.

Index Ticker	Robust CVaR portfolio	Benchmark "$1/n$" portfolio	Max or Min position
OEX	0.167	0.200	
SPX	0.350	0.200	$*$
MID	0.156	0.200	
RUT	0.128	0.200	
TYX	0.199	0.200	

To show that the Robust CVaR portfolio allocation model presented in Section 2 is robust in terms of its sensitivity to the input parameters (mainly option prices), we compute by how much can the model's input parameters be changed (independently), without changing any of the positions of the optimal portfolio by more than 1%. The corresponding results are presented in Table 4. From Table 4, it follows that most of the parameters can be substantially changed without affecting the optimal allocation (percentages listed with an "$*$" could still be increased). The lack of this characteristic is one of the main problems in classical portfolio allocation models (see, e.g., [5, 10, 13]). Furthermore, notice that since the model's parameters are not statistically estimated, but instead taken from current market prices, there is no need to consider estimation errors. The allowable increases and decreases in Table 4 indicate that the exact option price taken (say within its bid and ask price) or even the exact day (say within a couple of days) in which the price is taken is not going to affect significantly the optimal portfolio allocation.

As mentioned earlier, the worst-case approach of our Robust CVaR model aims not only at being robust in terms of the sensitivity to the model's parameters, but also at being robust in terms of the out-of-sample performance of the portfolio (as defined, e.g., in [12]). Our experiments above illustrate the first kind of robustness. Although showing the later kind of robustness requires performing comprehensive tests (see, e.g., [12]) that are outside the scope of this article, here we report the encouraging results of one test. Namely, we use out-of-sample data about the index prices from December 1st, 2004, to March 18th, 2005, to compute the out-of-sample Sharpe ratio of the daily portfolio returns (cf. [12]) in this period, for both the Robust CVaR, and the "$1/n$" portfolios. The Sharpe ratio of the Robust CVaR portfolio turns out to be 0.015, while the Sharpe ratio of the "$1/n$" portfolio turns out to be 0.012. The out-of-sample Sharpe ratio of the portfolio returns is used in [12] to measure the robustness of a portfolio (the higher the Sharpe ratio, the more robust the portfolio). In [12], it is shown that the myopic "$1/n$" portfolio has consistently higher out-of-sample Sharpe ratios than many classical portfolio allocation models. Thus, the fact that the Sharpe ratios of the Robust CVaR portfolio, and the "$1/n$" portfolio in Table 4 turn out to be comparable is encouraging towards showing the robustness in terms of out-of-sample performance of the Robust CVaR portfolio presented here.

Table 4: Sensitivity of Robust CVaR portfolio in Table 3 to (independent) changes in the option prices in Table 2. For each index, the first row lists the strike prices of the options corresponding to the option prices in Table 3. The second and third row indicate the percentage by which the corresponding option price in Table 3 can be increased and decreased without changing any of the positions of the portfolio by more than 1%. Entries with an "$*$" indicate that further increase or decrease is possible.

OEX	0	555	560	565	570	575	580	590	595	600
Increase	0.8%	4.4%	1.4%	3.0%	0.8%	7.0%	8.4%	10.2%*	5.0%	10.2%*
Decrease	1.6%	2.2%	4.4%	1.0%	7.4%	1.8%	10.2%*	7.0%	10.0%*	10.2%*

SPX	0	700	1175	1180	1200	1225	1235	1250	1275	1300
Increase	1.0%	3.8%	3.2%	1.4%	10.2%*	10.2%*	8.8%	10.2%*	10.2%*	10.2%*
Decrease	1.8%	1.6%	1.6%	3.6%	10.0%*	10.0%*	10.0%*	10.2%*	10.2%*	10.0%*

MID	0	630	640							
Increase	1.8%	7.8%	10.0%*							
Decrease	5.4%	10.2%*	10.2%*							

RUT	0	650	660	670	680					
Increase	9.6%	10.0%*	5.8%	7.8%	10.0%*					
Decrease	10.0%*	7.8%	10.2%*	10.0%*	10.0%*					

TYX	0	45	53							
Increase	10.0%*	7.2%	10.0%*							
Decrease	0.8%	10.0%*	10.0%*							

4 Extensions

In this section, we discuss extensions of the model; such as allowing short (instead of only long) positions, the possible use of American option prices (instead of European option prices), and adding a minimum expected return constraint.

4.1 Allowing short positions

Although, most asset allocation problems are constrained to have only long positions (see, e.g., [21]), in many instances, practitioners are interested in portfolios containing both long and short positions. We can easily extend our robust portfolio allocation model in order to consider both long and short positions, while maintaining the properties of the model. For that purpose, we use the following generalization of Theorem 1.

Theorem 2 ([28]) *Let $\mathcal{P}$ be given by (6), and $x \in \mathrm{IR}^n$. Let $x_{+,i} = \max\{x_i, 0\}$, and $x_{-,i} = \max\{-x_i, 0\}$, $i = 1, \dots, n$. If the arbitrage-free condition (7) holds, then $\sup_{\pi(S_T)\in\mathcal{P}} \mathbb{E}_{\pi(S_T)}((\hat{s}^{\mathrm{T}}x - (1-\alpha))^+)$ is equal to*

$$\begin{array}{lll} \min & y & \\ \text{s.t.} & y \geq x_+^{\mathrm{T}}\nu(\tau_{ij}) + x_-^{\mathrm{T}}\widetilde{\nu}(\tau_{ij}) - \tau_{ij}(1-\alpha), & i = 1, \dots, n, \quad j = 1, \dots, m \\ & y \geq x_+^{\mathrm{T}}\nu(0) + x_-^{\mathrm{T}}\widetilde{\nu}(0) & \\ & y \geq x_+^{\mathrm{T}}\nu(1) + x_-^{\mathrm{T}}\widetilde{\nu}(1) - (1-\alpha) & \\ & x = x_+ - x_- & \\ & y \in \mathrm{IR}, x_+, x_- \in \mathrm{IR}_+^n & \end{array} \tag{19}$$

where $\nu(\tau)_i$ is given by (15), $\widetilde{\nu}(\tau)_i$ is given by

$$\widetilde{\nu}(\tau)_i = \left(\frac{1}{S_{0,i}}\right)\left(p_i^0 - \min_{j=0,\dots,m}\left\{p_i^j + \tau K_i^j\right\}\right), \tag{20}$$

for $i = 1, \dots, n$; and τ_{ij} is given by (16), for $i = 1, \dots, n$, $j = 1, \dots, m$.

Proof. Follows from [28, Theorem 13, and Remark 15]. □

Using Theorem 2, and putting together (8) and (14), we obtain a LP formulation of the robust portfolio allocation model when both long and short positions are allowed. We state this LP formulation in the following proposition.

Proposition 3 *Let $\mathcal{P}$ be given by (6). If the arbitrage-free condition (7) holds, then solving the robust portfolio optimization model*

$$\begin{array}{lll} \max\limits_{x\in\mathcal{X}\subseteq\mathrm{IR}^n} & \sup\limits_{\pi(S_T)\in\mathcal{P}} & \mathrm{CVaR}_{\beta,\pi(S_T)}(1-\hat{s}^{\mathrm{T}}x) \\ \text{s.t.} & & e^{\mathrm{T}}x = 1, \end{array}$$

is equivalent to solving the following linear program

$$\begin{array}{lll} \min & y - \hat{p}^{\mathrm{T}}x - \beta\alpha & \\ \text{s.t.} & y \geq x_+^{\mathrm{T}}\nu(\tau_{ij}) + x_-^{\mathrm{T}}\widetilde{\nu}(\tau_{ij}) - \tau_{ij}(1-\alpha), & i = 1, \dots, n, \quad j = 1, \dots, m \\ & y \geq x_+^{\mathrm{T}}\nu(0) + x_-^{\mathrm{T}}\widetilde{\nu}(0) & \\ & y \geq x_+^{\mathrm{T}}\nu(1) + x_-^{\mathrm{T}}\widetilde{\nu}(1) - (1-\alpha) & \\ & x = x_+ - x_- & \\ & e^{\mathrm{T}}x = 1 & \\ & x \in \mathrm{IR}_+^n, x_+, x_- \in \mathrm{IR}_+^n, y \in \mathrm{IR}, \alpha \in \mathrm{IR} & \end{array} \tag{21}$$

where $\nu(\tau)_i$ is given by (15), $\widetilde{\nu}(\tau)_i$ is given by (20), for $i = 1, \dots, n$; and τ_{ij} is given by (16), for $i = 1, \dots, n$, $j = 1, \dots, m$.

Proof. Notice that since $\nu(\cdot), \widetilde{\nu}(\cdot) \geq 0$, it follows that in any optimal solution of (21), $x_{+,i} = \max\{x_i, 0\}$, and $x_{-,i} = \max\{-x_i, 0\}$, $i = 1, \dots, n$. Thus, the result follows from Theorem 2. □

Proposition 3 clearly shows that the properties of the robust portfolio model discussed so far, are maintained when both long and short positions are considered in the portfolio.

4.2 Using American option prices

While forward (future) options by definition give the right to obtain an asset at a specified future date, call options are typically American options; that is, the right to buy the asset at the specified strike price can be exercised at any time between the purchase of the option and the maturity of the option. This is especially common for equities. This is the reason why we consider index options; which are European options, in our experiments of Section 3. However, there is no empirical reason to forbid the use of American option prices instead of European option prices. In fact, this is commonly done in the *arbitrage bounds* literature (see, e.g., [11, 17, 28] and the references therein); which is at the basis of our robust portfolio allocation model. Also, many American options can be considered as European, since they are typically not exercised prior to maturity. More importantly, as we will see next, using American option prices instead of European option prices, only leads to a more robust portfolio allocation model.

Let $\widetilde{p}_i^j$ be the price of an American call option on asset i with strike K_i^j maturing at $t = T$, for $j = 1, \ldots, m$, $i = 1, \ldots, n$ (recall (6)). Also let $\widetilde{\mathcal{P}}$ be the uncertainty set obtained when replacing $p_i^j \to \widetilde{p}_i^j$ in (6) for any subset of indices $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$. The following claim follows from convex duality and the fact that $p_i^j \leq \widetilde{p}_i^j$, for $j = 1, \ldots, m$, $i = 1, \ldots, n$ (see, e.g., [28, Section 3] for details).

Claim 1 *If the option prices in both $\mathcal{P}$, and $\widetilde{\mathcal{P}}$ satisfy the arbitrage condition* (7)*, then*

$$\sup_{\pi(S_T)\in\mathcal{P}} \mathbb{E}_{\pi(S_T)}((\hat{s}^{\mathrm{T}}x - (1-\alpha))^+) \leq \sup_{\pi(S_T)\in\widetilde{\mathcal{P}}} \mathbb{E}_{\pi(S_T)}((\hat{s}^{\mathrm{T}}x - (1-\alpha))^+).$$

From Claim 1 and (14), it clearly follows that

$$\mathrm{WCVaR}_{\beta,\mathcal{P}}(1-\hat{s}^{\mathrm{T}}x) \leq \mathrm{WCVaR}_{\beta,\widetilde{\mathcal{P}}}(1-\hat{s}^{\mathrm{T}}x), \tag{22}$$

where we have made explicit the dependence of the WCVaR on the uncertainty set $\mathcal{P}$, when using European option prices; and $\widetilde{\mathcal{P}}$, when using American option prices. Recall that WCVaR is related to the expected value of the β-quantile of the portfolio loss distribution. Therefore, from (22) it follows that by using American option prices we are using a more pessimistic estimate of the measure of risk; which makes the model (8) more robust.

To illustrate the use of American option prices, consider an experiment similar to the one done in Section 3, but this time we are interested in rebalancing a benchmark "$1/n$" portfolio on the assets comprising the Dow Jones 30 Index. We assume that the allocation of funds is being made on May 17th, 2004, and the maturity date of the options is selected to be June 18th, 2004. The corresponding option prices and asset prices data is presented in Table 5 (source: WRDS. This data is similar to the one used in [11, 17, 28]). Using this data, we obtain the rebalanced portfolio shown in Table 6.

4.3 Minimum expected return constraint

Most of the portfolio allocation models that look for a portfolio with minimum risk include in their formulation a constraint on the minimum expected return. In our model, we can include this type of constraint by starting (instead of (4)) with the following nominal portfolio allocation model.

$$\begin{array}{cl} \min\limits_{x\in\mathcal{X}\subseteq\mathbb{R}^n} & \mathrm{CVaR}_\beta(1-\hat{s}^{\mathrm{T}}x) \\ \text{s.t.} & \mathbb{E}(\hat{s}^{\mathrm{T}}x - 1) \geq \mu_o \\ & e^{\mathrm{T}}x = 1, \end{array} \tag{23}$$

for a given minimum expected return μ_o. Notice that as in Section 2, we have expressed the portfolio return in terms of the asset prices at maturity. To construct a robust version of (23), we minimize the worst-case portfolio CVaR over all assets price distributions at maturity that *replicate* current prices of European forward and call options on the assets. Additionally, we require that the worst-case portfolio expected return be greater than μ_o. Specifically, we introduce the following robust formulation of (23):

$$\begin{array}{ccl} \min\limits_{x\in\mathcal{X}\subseteq\mathbb{R}^n} & \sup\limits_{\pi(S_T)\in\mathcal{P}} & \mathrm{CVaR}_{\beta,\pi(S_T)}(1-\hat{s}^{\mathrm{T}}x) \\ \text{s.t.} & & \inf\limits_{\pi(S_T)\in\mathcal{P}} \mathbb{E}(\hat{s}^{\mathrm{T}}x - 1) \geq \mu_o \\ & & e^{\mathrm{T}}x = 1. \end{array} \tag{24}$$

Notice that for any $\pi(S_T) \in \mathcal{P}$, $\mathbb{E}(\hat{s}^{\mathrm{T}}x - 1) = \hat{p}^{\mathrm{T}}x - 1$ (recall (6) and (13)). Thus, the robust model (24) is equivalent to:

$$\begin{array}{ccl} \min\limits_{x\in\mathcal{X}\subseteq\mathbb{R}^n} & \sup\limits_{\pi(S_T)\in\mathcal{P}} & \mathrm{CVaR}_{\beta,\pi(S_T)}(1-\hat{s}^{\mathrm{T}}x) \\ \text{s.t.} & & \hat{p}^{\mathrm{T}}x - 1 \geq \mu_o \\ & & e^{\mathrm{T}}x = 1. \end{array} \tag{25}$$

Therefore, all the analysis and results presented through out the paper will follow when we consider a constraint on the minimum expected return of the portfolio, simply by adding the linear constraint $\hat{p}^{\mathrm{T}}x - 1 \geq \mu_o$. In particular, the LP formulation of the robust model (25) is:

$$\begin{array}{ll} \min & y - \hat{p}^{\mathrm{T}}x - \beta\alpha \\ \text{s.t.} & y \geq x_+^{\mathrm{T}}\nu(\tau_{ij}) + x_-^{\mathrm{T}}\widetilde{\nu}(\tau_{ij}) - \tau_{ij}(1-\alpha), \qquad i = 1,\ldots,n, \quad j = 1,\ldots,m \\ & y \geq x_+^{\mathrm{T}}\nu(0) + x_-^{\mathrm{T}}\widetilde{\nu}(0) \\ & y \geq x_+^{\mathrm{T}}\nu(1) + x_-^{\mathrm{T}}\widetilde{\nu}(1) - (1-\alpha) \\ & x = x_+ - x_- \\ & \hat{p}^{\mathrm{T}}x - 1 \geq \mu_o \\ & e^{\mathrm{T}}x = 1 \\ & x \in \mathbb{R}_+^n, x_+, x_- \in \mathbb{R}_+^n, y \in \mathbb{R}, \alpha \in \mathbb{R} \end{array} \tag{26}$$

where $\nu(\tau)_i$ is given by (15), $\widetilde{\nu}(\tau)_i$ is given by (20), for $i = 1,\ldots,n$; and τ_{ij} is given by (16), for $i = 1,\ldots,n$, $j = 1,\ldots,m$.

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Table 5: American option prices from May 17th, 2004 on contracts expiring on June 18th, 2004. The table gives forward prices and mid-market prices on American call options on May 17th, 2004 for the assets comprising the Dow Jones 30 Index. For every asset, the first row corresponds to the different strike prices, and the second row corresponds to the option prices. The entry 0.00 for each asset gives the forward option price. The price below each asset corresponds to the asset price (value) on May 17th, 2004.

Asset Ticker	Option prices and strikes								
MCD	0.00	20.00	25.00	27.50	30.00				
25.87	25.00	5.90	1.35	0.30	0.05				
AA	0.00	25.00	27.50	30.00	32.50	35.00			
28.70	25.88	4.00	2.13	0.90	0.25	0.10			
AIG	0.00	65.00	70.00	75.00					
69.39	59.30	5.35	2.05	0.43					
AXP	0.00	47.50	50.00						
48.77	38.94	2.13	0.75						
BA	0.00	40.00	42.50	45.00					
42.73	36.60	3.00	1.30	0.35					
VZ	0.00	35.00	37.50						
35.78	26.65	1.45	0.35						
CAT	0.00	60.00	70.00	75.00	80.00	85.00			
73.54	70.25	13.70	4.85	1.92	0.55	0.15			
DD	0.00	40.00	42.50	45.00					
41.19	32.10	1.90	0.63	0.13					
DIS	0.00	20.00	22.50	25.00	27.50				
22.90	20.98	2.98	0.98	0.17	0.05				
GE	0.00	25.00	27.50	30.00	32.50	35.00	37.50		
29.97	29.25	5.00	2.65	0.80	0.13	0.03	0.03		
WMT	0.00	47.50	50.00	55.00	60.00	65.00	70.00		
54.70	52.90	7.30	5.00	1.38	0.13	0.03	0.03		
GM	0.00	35.00	40.00	42.50	45.00	47.50	50.00	55.00	
43.40	42.07	8.65	4.10	2.25	1.00	0.35	0.13	0.03	
HD	0.00	30.00	32.50	35.00	37.50	40.00			
33.47	30.25	3.70	1.77	0.57	0.13	0.03			
HON	0.00	30.00	32.50	35.00	37.50	40.00			
32.60	27.98	2.78	1.08	0.25	0.05	0.03			
HPQ	0.00	15.00	17.50	20.00	22.50				
19.50	18.95	4.55	2.25	0.68	0.13				
IBM	0.00	80.00	85.00	90.00	95.00	100.00			
85.53	75.20	6.20	2.58	0.68	0.17	0.03			
JPM	0.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	45.00
35.31	34.85	7.90	5.50	3.20	1.40	0.40	0.08	0.05	0.03
KO	0.00	42.50	47.50	50.00	55.00				
49.81	48.68	7.35	2.63	0.93	0.03				
XOM	0.00	40.00	42.50	45.00					
43.05	38.50	3.30	1.40	0.35					
INTC	0.00	20.00	22.50	25.00	27.50	30.00			
26.84	26.65	6.85	4.40	2.27	0.75	0.15			
JNJ	0.00	50.00	55.00						
54.70	49.02	4.90	1.08						
UTX	0.00	80.00	85.00	90.00					
82.34	62.30	3.50	1.15	0.25					
MMM	0.00	80.00	85.00	90.00					
83.22	67.10	4.10	1.23	0.20					
MO	0.00	45.00	47.50	50.00	55.00	60.00			
49.47	45.98	4.80	2.72	1.17	0.13	0.05			
MRK	0.00	45.00	47.50	50.00					
46.64	37.15	2.05	0.65	0.13					
PFE	0.00	30.00	35.00	37.50	40.00	42.50			
35.50	33.65	5.60	1.25	0.28	0.08	0.03			
PG	0.00	90.00	95.00	100.00	105.00	110.00	115.00		
106.28	105.05	16.40	11.55	7.00	3.20	0.95	0.23		
SBC	0.00	15.00	25.00						
24.30	23.49	9.30	0.38						
MSFT	0.00	20.00	22.50	25.00	27.50				
25.54	25.40	5.60	3.15	1.10	0.17				
C	0.00	30.00	35.00	40.00	42.50	45.00	47.50	50.00	55.00
44.86	44.75	14.90	9.95	5.15	2.95	1.27	0.38	0.10	0.03

Table 6: Rebalancing a Dow Jones 30 benchmark "$1/n$" portfolio using the Robust CVaR portfolio allocation model (17) with $\beta = 0.96$, $\mathcal{X}$ as in (18), and maximum rebalancing of 5% ($\delta = 1.5$, $\bar{x} = (1/30)e$ in (18)). The last column indicates if the rebalancing has reached the maximum possible.

Asset Ticker	Robust CVaR portfolio	Benchmark "$1/n$" portfolio	Max or Min position
MCD	5.66%	3.33%	
AA	0.45%	3.33%	
AIG	0.34%	3.33%	
AXP	0.34%	3.33%	
BA	0.36%	3.33%	
VZ	0.35%	3.33%	
CAT	2.67%	3.33%	
DD	0.34%	3.33%	
DIS	0.79%	3.33%	
GE	8.33%	3.33%	*
WMT	7.80%	3.33%	
GM	5.64%	3.33%	
HD	0.55%	3.33%	
HON	0.36%	3.33%	
HPQ	7.64%	3.33%	
IBM	0.42%	3.33%	
JPM	8.33%	3.33%	*
KO	8.33%	3.33%	*
XOM	0.55%	3.33%	
INTC	8.33%	3.33%	*
JNJ	0.60%	3.33%	
UTX	0.35%	3.33%	
MMM	0.34%	3.33%	
MO	1.22%	3.33%	
MRK	0.34%	3.33%	
PFE	2.13%	3.33%	
PG	8.33%	3.33%	*
SBC	2.42%	3.33%	
MSFT	8.33%	3.33%	*
C	8.33%	3.33%	*